ALGONQUIN COLLEGE

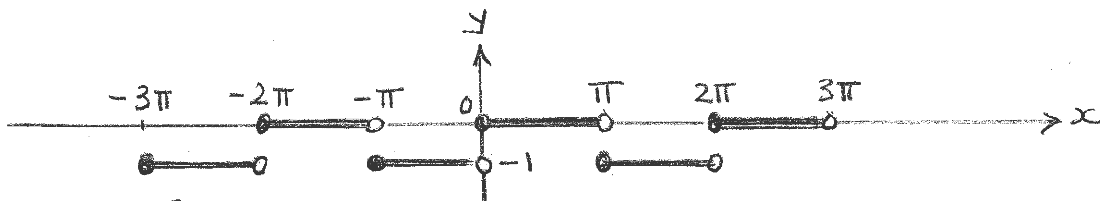
Section:__________ Name: Solution

Lab 14 - Fourier Series

1. Given the periodic functions below.
   - **i.** Sketch their graphs over three periods.
   - **ii.** Find their Fourier coefficients a_0, a_1, a_2, b_1, b_2.
   - **iii.** Use the Fourier coefficients found in **ii**. to write the corresponding Fourier series.

a. $f(x) = \begin{cases} -1 & \text{for } -\pi \le x < 0 \\ 0 & \text{for } 0 \le x < \pi, \quad \text{period} = 2\pi \end{cases}$

i.



ii. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -1\,dx$

$= \frac{1}{2\pi}\left[-x\right]_{-\pi}^{0}$

$= \frac{1}{2\pi}\left\{[0]-[\pi]\right\} = -\frac{1}{2}$

$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 \cdot \cos x\,dx =$

$= \frac{1}{\pi}\left[-\sin x\right]_{-\pi}^{0}$

$= \frac{1}{\pi}\left\{[0]-[0]\right\} = 0$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 . \cos 2x\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}\sin 2x\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left\{[0] - [0]\right\} = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 . \sin x\,dx$$

$$= \frac{1}{\pi}\left[\cos x\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left\{[1] - [-1]\right\} = \frac{2}{\pi}$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1 . \sin 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\cos 2x\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left\{\left[\frac{1}{2}\right] - \left[\frac{1}{2}\right]\right\} = 0$$

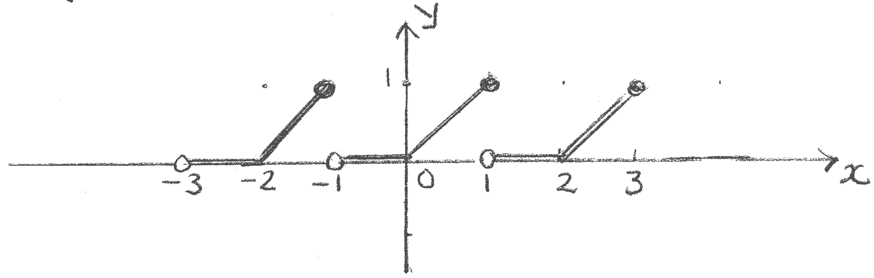
iii. $f(x) = a_0 + a_1\cos x + a_2\cos 2x + \dots$
$\quad + b_1\sin x + b_2\sin 2x + \dots$

$= -\frac{1}{2} + 0 + 0 + \dots$
$\quad + \frac{2}{\pi}\sin x + 0 + \dots$

$$= -\frac{1}{2} + \dots + \frac{2}{\pi}\sin x + \dots$$

b. $f(x) = \begin{cases} 0 & \text{for } -1 < x \le 0 \\ x & \text{for } 0 < x \le 1, \quad \text{period} = 2 \end{cases}$

i.



ii $P = 2L \Rightarrow 2 = 2L \Rightarrow L = 1$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_0^1 x\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{2}\left\{\left[\frac{1}{2}\right] - [0]\right\} = \frac{1}{4}$$

$$a_1 = \frac{1}{L}\int_{-L}^{L} f(x)\cos x\,dx = \int_0^1 x\cos x\,dx$$

$$= \left[\frac{1}{\pi^2}\cos\pi x + \frac{1}{\pi}x\sin\pi x\right]_0^1$$

$$= \left[-\frac{1}{\pi^2}\right] - \left[\frac{1}{\pi^2}\right] = -\frac{2}{\pi^2}$$

$$a_2 = \frac{1}{L}\int_{-L}^{L} f(x)\cos 2\pi x\,dx = \int_0^1 x\cos 2\pi x\,dx$$

$$= \left[\frac{1}{4\pi^2}\cos 2\pi x + \frac{1}{2\pi}x\sin 2\pi x\right]_0^1$$

$$= \left[\frac{1}{4\pi^2}\right] - \left[\frac{1}{4\pi^2}\right] = 0$$

$$b_1 = \frac{1}{L}\int_{-L}^{L} f(x)\sin\pi x\,dx = \int_0^1 x\sin\pi x\,dx$$

$$= \left[\frac{1}{\pi^2}\sin\pi x - \frac{1}{\pi}x\cos\pi x\right]_0^1$$

$$= \left[\frac{1}{\pi}\right] - \left[0\right] = \frac{1}{\pi}$$

$$b_2 = \frac{1}{L}\int_{-L}^{L} f(x)\sin 2\pi x\,dx = \int_0^1 x\sin 2\pi x\,dx$$

$$= \left[\frac{1}{4\pi^2}\sin 2\pi x - \frac{1}{2\pi}x\cos 2\pi x\right]_0^1$$

$$= \left[-\frac{1}{2\pi}\right] - \left[0\right] = -\frac{1}{2\pi}$$

iii. $$f(x) = a_0 + a_1\cos\frac{\pi x}{L} + a_2\cos\frac{2\pi x}{L} + \cdots$$

$$+ b_1\sin\frac{\pi x}{L} + a_2\sin\frac{2\pi x}{L} + \cdots$$

$$= \frac{1}{4} - \frac{2}{\pi^2}\cos\pi x + 0 \quad + \cdots$$

$$+ \frac{1}{\pi}\sin\pi x - \frac{1}{2\pi}\sin 2\pi x + \cdots$$

$$= \frac{1}{4} - \frac{2}{\pi^2}\cos\pi x + \cdots + \frac{1}{\pi}\sin\pi x - \frac{1}{2\pi}\sin 2\pi x + \cdots$$