

Lab 8 - Laplace Transforms

1. Find the Laplace transforms of the following functions.

a. $f(t) = 1 - \sin 2t$

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(1 - \sin 2t) \\ &= \mathcal{L}(1) - \mathcal{L}(\sin 2t) \\ &= \frac{1}{s} - \frac{2}{s^2 + 4} \\ &= \frac{s^2 - 2s + 4}{s(s^2 + 4)} \end{aligned}$$

b. $f(t) = 5t^3e^{-t} - t \cos \sqrt{5}t$

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(5t^3e^{-t} - t \cos \sqrt{5}t) \\ &= 5 \mathcal{L}(t^3e^{-t}) - \mathcal{L}(t \cos \sqrt{5}t) \\ &= 5 \cdot \frac{3!}{(s+1)^4} - \frac{s^2 - 5}{(s^2 + 5)^2} \\ &= \frac{30}{(s+1)^4} - \frac{s^2 - 5}{(s^2 + 5)^2} \end{aligned}$$

2. Find the inverse Laplace transforms of the following functions.

a. $F(s) = \frac{s+4}{(s-1)(s+3)}$

Step 1:
$$\frac{s+4}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

Step 2: $s+4 = A(s+3) + B(s-1)$ (*)

Step 3: * Find A: $s=1$ in (*) $\Rightarrow 5 = 4A \Rightarrow A = 5/4$

* Find B: $s=-3$ in (*) $\Rightarrow 1 = -4B \Rightarrow B = -1/4$

$\therefore \frac{s+4}{(s-1)(s+3)} = A \cdot \frac{1}{s-1} + B \cdot \frac{1}{s+3}$

$$\frac{s+4}{(s-1)(s+3)} = \frac{5}{4} \cdot \frac{1}{s-1} - \frac{1}{4} \cdot \frac{1}{s+3}$$

Step 4: $\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left[\frac{s+4}{(s-1)(s+3)}\right]$

$$\mathcal{L}^{-1}(F) = \frac{5}{4} \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{1}{4} \mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$

$$\mathcal{L}^{-1}(F) = \frac{5}{4} e^t - \frac{1}{4} e^{-3t}$$

b. $F(s) = \frac{3}{s^2(s-1)}$

Step 1:
$$\frac{3}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$$

Step 2:
$$3 = A s (s-1) + B (s-1) + C s^2 \quad (*)$$

Step 3: * Find C: $s=1$ in (*) $\Rightarrow 3 = C$

* Find B: $s=0$ in (*) $\Rightarrow 3 = -B \Rightarrow B = -3$

* Find A: $s=2, C=3, B=-3$ in (*) \Rightarrow

$$3 = 2A - 3 + 12 \Rightarrow A = -3$$

$$\therefore \frac{3}{s^2(s-1)} = -3 \cdot \frac{1}{s} - 3 \cdot \frac{1}{s^2} + 3 \cdot \frac{1}{s-1}$$

Step 4:
$$\mathcal{L}^{-1}(F) = -3 \mathcal{L}^{-1}\left(\frac{1}{s}\right) - 3 \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 3 \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$\mathcal{L}^{-1}(F) = -3 - 3t + 3e^t$$