ALGONQUIN COLLEGE

Section:__________ Name: Solution

Lab 9 - Solving Differential Equations by Laplace Transforms

1. Solve the following differential equations using Laplace transforms, where the function is subject to the given conditions.
$y'' - 2y' + y = e^{2t}, \quad y(0) = 1, y'(0) = 3$

Step 1: $\mathcal{L}(y'') - 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^{2t})$

$s^2\mathcal{L}(y) - sy(0) - y'(0) - 2[s\mathcal{L}(y) - y(0)] + \mathcal{L}(y) = \mathcal{L}(e^{2t})$

$Y = \mathcal{L}(y), \; y(0) = 1, \; y'(0) = 3, \; \mathcal{L}(e^{2t}) = \frac{1}{s-2} \Rightarrow$

$s^2Y - s - 3 - 2sY + 2 + Y = \frac{1}{s-2} \Rightarrow$

$Y = \frac{s^2 - s - 1}{(s-2)(s-1)^2}$

Step 2:

* $\frac{s^2 - s - 1}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$
* $s^2 - s - 1 = A(s-1)^2 + B(s-2)(s-1) + C(s-2)$ (*)
* Find A: $s = 2$ in (*) $\Rightarrow A = 1$
* Find C; $s = 1$ in (*) $\Rightarrow C = 1$
* Find B: $s = 0$, $A = 1$, $C = 1$ in (*) $\Rightarrow B = 0$

$\therefore \frac{s^2 - s - 1}{(s-2)(s-1)^2} = \frac{1}{s-2} + \frac{1}{(s-1)^2}$

Step 3:

$y = \mathcal{L}^{-1}(Y)$

$y = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]$

$y = e^{2t} + te^{t}$

2. Recall: The impressed voltage in an electric circuit equals the sum of the voltages across the components of the circuit. For a circuit with a resistance R (in ohms), an inductance L (in henrys), a capacitance C (in farads), and a voltage source E (in volts), we have

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

where q represents the electric charge (in coulombs) and t represents the time (in seconds).

A $50\ \Omega$ resistor, a $400\ \mu F$ capacitor, and an $8\ V$ battery are connected in series. Find the charge on the capacitor as a function of time t if the initial charge is zero.

$L = 0\,H \quad R = 50\,\Omega \quad C = 400\,\mu F = 0.000\,04\,F \quad E = 8V$

$$50q' + \frac{1}{0.000\,04}q = 8 \Rightarrow 25q' + 12\,500\,q = 4 \quad ; q(0) = 0$$

Step 1:

$$25\mathcal{L}(q') + 12\,500\,\mathcal{L}(q) = \mathcal{L}(4) \Rightarrow$$

$$25[s\mathcal{L}(q) - q(0)] + 12\,500\,\mathcal{L}(q) = \mathcal{L}(4).$$

$$Q = \mathcal{L}(q),\ q(0) = 0,\ \mathcal{L}(4) = \frac{4}{s} \Rightarrow$$

$$25sQ + 12\,500\,Q = \frac{4}{s} \Rightarrow Q = \frac{4}{25}\cdot\frac{1}{s(s+500)}$$

Step 2:

$$\frac{1}{s(s+500)} = \frac{1}{500}\cdot\frac{1}{s} - \frac{1}{500}\cdot\frac{1}{s+500}$$

$$\therefore\ Q = \frac{4}{25}\cdot\left[\frac{1}{500}\cdot\frac{1}{s} - \frac{1}{500}\cdot\frac{1}{s+500}\right]$$

$$= \frac{1}{3\,125}\cdot\frac{1}{s} - \frac{1}{3\,125}\cdot\frac{1}{s+500}$$

Step 3:

$$q = \mathcal{L}^{-1}(Q)$$

$$q = \frac{1}{3\,125}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{3\,125}\cdot\mathcal{L}^{-1}\left(\frac{1}{s+500}\right)$$

$$q = \frac{1}{3\,125} - \frac{1}{3\,125}e^{-500t}$$

OR

$$q = 3.2\times10^{-4} - 3.2\times10^{-4}e^{-500t}$$