MAT8103 – Assignment 2 Name: Solutions

Due date: ____ 52

1) When working with higher order linear differential equations we need to solve an auxiliary equation to get the complimentary solution. Complete the following table for either a possible original differential equation or its associated auxiliary equation. **(1 mark each)**

One Possible Differential Equation	Auxiliary Equation
$6\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = xe^{5x}$	$6m^2 - 7m + 2 = 0$
$2D^7y - 7D^5y + 9D^2y - y = 0$	$2m^7 - 7m^5 + 9m^2 - 1 = 0$
$2D^3y - 12D^2y + 12Dy - 8y = f(x)$	$2m^3 - 12m^2 + 12m - 8 = 0$
$D^4y - 5D^3y + 2Dy - 3y = e^{-2x}\sin(x)$	$m^4 - 5m^3 + 2m - 3 = 0$
$D^5y - 2D^4y + 4D^3y - 8D^2y = f(x)$	$m^5 - 2m^4 + 4m^3 - 8m^2 = 0$

2) When working with constant coefficient homogeneous linear differential equations, we can find the solutions to the auxiliary equations and then simply write the answers based on the roots. Fill in the following table for either the equation solutions or the roots of the auxiliary equation. **(2 marks each)**

Roots of the Auxiliary Equation	Differential Equation Complimentary Solutions
$m = -1, 2$	$y = c_1e^{-x} + c_2e^{2x}$
$m = 5, 5$	$y = C_1e^{5x} + C_2xe^{5x}$
$m = 4 \pm 7j$	$y = e^{4x}(c_1\cos(7x) + c_2\sin(7x))$
$m = -6, -6, -6, 2$	$y = C_1e^{-6x} + C_2xe^{-6x} + C_3x^2e^{-6x} + C_4e^{2x}$
$m = -1, 2, 2j$	$y = c_1e^{-x} + c_2e^{2x} + c_3\cos(2x) + c_4\sin(2x)$
$m = -3j, -3j$	$y = C_1\cos(-3x) + C_2\sin(-3x) + C_3x\cos(-3x) + C_4x\sin(-3x)$
$m = -4, 1, 3, 3$	$y = c_1e^{-4x} + c_2e^{x} + (c_3 + c_4x)e^{3x}$

3) While solving the auxiliary equation is the key to finding the complimentary solution, particular solutions are found by using the method of undetermined coefficients. For each differential equation below, solve the auxiliary equation for the complimentary solution and then state your guess at the particular solution, but **do not** solve for the coefficients. **(4 marks each)**

Differential Equation	Complimentary and Particular Solutions
$D^2y - Dy - 6y = 6x^2 + xe^{3x}$ $m^2 - m - 6 = 0$ $(m+2)(m-3) = 0$	$y_c = c_1e^{-2x} + c_2e^{3x}$ $y_p = A + Bx + Cx^2 + Dxe^{3x} + Ex^2e^{3x}$
$5D^2y - 6Dy + 3y = 2 - 3x^2$ $5m^2 - 6m + 3 = 0$ $m = \frac{6 \pm \sqrt{36-60}}{10} = \frac{6 \pm j\sqrt{24}}{10} = \frac{6 \pm 2j\sqrt{6}}{10} = \frac{3}{5} \pm j\frac{\sqrt{6}}{5}$	$y_c = e^{3/5 x}\left(c_1\cos\left(\frac{\sqrt{6}}{5}x\right) + c_2\sin\left(\frac{\sqrt{6}}{5}x\right)\right)$ $y_p = A + Bx + Cx^2$
$D^2y + 2Dy + y = 3\cos(x) - \sin(x)$ $m^2 + 2m + 1 = 0$ P:1 S:2 1 & 1 $(m+1)^2 = 0$ $m = -1$ repeated	$y_c = c_1e^{-x} + c_2xe^{-x}$ $y_p = A\cos x + B\sin x$
$D^2y - 2Dy - 8y = 7 - e^{4x}$ $m^2 - 2m - 8 = 0$ P:−8 S:−2 −4 & 2 $(m-4)(m+2) = 0$ $m = 4$ $m = -2$	$y_c = c_1e^{4x} + c_2e^{-2x}$ can't use this $y_p = A + Be^{4x}$ ‖ use $y_p = A + Bxe^{x}$ instead
$5D^4y - 4D^3y - D^2y = x^2 + 3e^x$ $5m^4 - 4m^3 - m^2 = 0$ $m^2(5m^2 - 4m - 1) = 0$ P:−5 S:−4 −5 & 1, $-\frac{5}{5}$ & $\frac{1}{5}$ $m = 0$ repeated $(5m+1)(m-1) = 0$ $m = -\frac{1}{5}$ $m = 1$	$y_c = c_1 + c_2x + c_3e^{-1/5 x} + c_4e^{x}$ can't use this $y_p = A + Bx + Cx^2 + De^x$ [look at Q4d) to see why this doesn't work!] $y_p = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fxe^x$

4) Complete the general solutions to the bottom 4 differential equations from the table in question 3). First substitute your particular solution into the differential equation and solve for your undetermined coefficients. Then state your general solution as the total of your complimentary and particular solutions. **(5 marks each)**

a) $5D^2y - 6Dy + 3y = 2 - 3x^2$ Soln: $y = y_c + y_p$

If $y_p = A + Bx + Cx^2$ then $y_p' = B + 2Cx$ & $y_p'' = 2C$ and

$5(2C) - 6(B + 2Cx) + 3(A + Bx + Cx^2) = 2 - 3x^2$

$(10C - 6B + 3A) + x(-12C + 3B) + x^2(3C) = 2 - 3x^2$

So $\left.\begin{array}{l} 10C - 6B + 3A = 2 \\ -12C + 3B = 0 \\ 3C = -3 \end{array}\right\}$ $C = -3/3 \Leftrightarrow C = -1$

$-12C + 3B = 0 \Leftrightarrow 12 + 3B = 0 \Leftrightarrow B = -4$

$10C - 6B + 3A = 2 = 2 \Leftrightarrow -10 + 24 + 3A = 2$

$\Leftrightarrow A = \frac{-12}{3} = -4$

Therefore, $y(x) = y_c + y_p = \boxed{e^{3/5x}\left(C_1\cos\left(\frac{\sqrt{6}}{5}x\right) + C_2\sin\left(\frac{\sqrt{6}}{5}x\right)\right) - 4 - 4x - x^2}$

b) $D^2y + 2Dy + y = 3\cos(x) - \sin(x)$

If $y_p = A\cos x + B\sin x$, then $y_p' = -A\sin x + B\cos x$ & $y_p'' = -A\cos x - B\sin x$ and

$(-A\cos x - B\sin x) + 2(-A\sin x + B\cos x) + (A\cos x + B\sin x) = 3\cos x - \sin x$

$\cos x(\cancel{-A} + 2B + \cancel{A}) + \sin x(\cancel{-B} - 2A + \cancel{B}) = 3\cos x - \sin x$

So $\left.\begin{array}{l} 2B = 3 \\ -2A = -1 \end{array}\right\}$ $\begin{array}{l} B = 3/2 \\ A = 1/2 \end{array}$ and $y_p = \frac{1}{2}\cos x + \frac{3}{2}\sin x$

Therefore, $y(x) = y_c + y_p = \boxed{C_1 e^{-x} + C_2 x e^{-x} + \frac{1}{2}\cos x + \frac{3}{2}\sin x}$

c) $D^2y - 2Dy - 8y = 7 - e^{4x}$

If $y_p = A + Be^{4x}$, then $y_p' = 4Be^{4x}$ and $y_p'' = 16Be^{4x}$ and

$$16Be^{4x} - 2(4Be^{4x}) - 8(A + Be^{4x}) = 7 - e^{4x}$$

$$-8A + e^{4x}\underbrace{(16B - 8B - 8B)}_{\text{this gives } 0B \Leftrightarrow \text{doesn't work since } e^{4x} \text{ is a solution in } y_c.} = 7 - e^{4x}$$

So $-8A = 7$

Instead, choose $y_p = A + Bxe^{4x}$, $y_p' = Be^{4x} + 4Bxe^{4x}$, $y_p'' = 4Be^{4x} + 4Be^{4x} + 16Bxe^{4x}$

and $4Be^{4x} + 4Be^{4x} + 16Bxe^{4x} - 2(Be^{4x} + 4Bxe^{4x}) - 8(A + Bxe^{4x}) = 7 - e^{4x}$

$-8A + 6Be^{4x} = 7 - e^{4x} \Leftrightarrow$ So $-8A = 7 \Leftrightarrow A = -7/8$

$6B = -1 \Leftrightarrow B = -1/6$

And $\boxed{y(x) = C_1e^{4x} + C_2e^{-2x} - \frac{7}{8} - \frac{1}{6}xe^{4x}}$

*This isn't correct!

d) $5D^4y - 4D^3y - D^2y = x^2 + 3e^x$

If $y_p = A + Bx + Cx^2 + De^x$, then $y_p' = B + 2Cx + De^x$ & $y_p'' = 2C + De^x$

& $y_p''' = De^x$ & $y_p'''' = De^x$ and

← can't use De^x since e^x is part of y_c

$$\underbrace{5De^x - 4De^x - (2C + De^x)}_{\text{we need an } x^2 \text{ term}} = x^2 + 3e^x$$

and we don't have it since y'' doesn't contain an x^2

Instead

$y_p = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fxe^x$

$y_p' = B + 2Cx + 3Dx^2 + 4Ex^3 + Fe^x + Fxe^x$

$y_p'' = 2C + 6Dx + 12Ex^2 + Fe^x + Fe^x + Fxe^x$ $\quad y_p'''' = 24E + 4Fe^x + Fxe^x$

$y_p''' = 6D + 24Ex + Fe^x + Fe^x + Fe^x + Fxe^x$

So

$$5(24E+4Fe^x+Fxe^x)-4(6D+24Ex+3Fe^x+Fxe^x)$$
$$-(2C+6Dx+12Ex^2+2Fe^x+Fxe^x)=x^2+3e^x$$

$$(120E-24D-2C)+x(-96E-6D)+x^2(-12E)$$
$$+e^x(20F-12F-2F)+\cancel{xe^x(5F-4F-F)}=x^2+3e^x$$

We have

$$\left.\begin{aligned}120E-24D-2C&=0\\ -96E-6D&=0\\ -12E&=1\\ 6F&=3\end{aligned}\right\}\quad \begin{aligned}&F=3/6=1/2\\ &E=-1/12\\ &-96(-1/12)-6D=0\\ &\Leftrightarrow D=8/6=4/3\end{aligned}$$

$$120(-1/12)-24(4/3)-2C=0$$
$$-10-32-2C=0$$
$$C=42/2=21$$

Therefore, $y_p=21x^2+\frac{4}{3}x^3-\frac{1}{12}x^4+\frac{1}{2}xe^x$

and the general solution is

$$\boxed{y(x)=C_1+C_2x+C_3e^{-1/5x}+C_4e^x+21x^2+\frac{4}{3}x^3-\frac{1}{12}x^4+\frac{1}{2}xe^x}$$