Laplace Transforms

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If $f(x,y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. For example, by holding y constant, we see $\int_1^2 xy^2 dx = 3y^2$. Similarly, a definite integral such as $\int_a^b K(s,t)f(t)dt$ transfers the function f of a variable t into a function F of the variable s. We are particularly interested in an **integral transform**, where the interval of integration is the unbounded interval $(0 \le t < \infty)$. If $f(t)$ is defined for $t \ge 0$, then the improper integral $\int_0^\infty K(s,t)f(t)dt$ is defined as a limit:

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty} \int_0^b K(s,t)f(t)dt \qquad (1)$$

If the limit in (1) exists, then we say that the integral exists or is **convergent;** if the limit does not exist, the integral does not exist and is **divergent.** The limit in (1) will, in general, exist for only certain values of the variable s.

The function $K(s,t)$ in (1) is called the **kernel** of the transform.

The choice $K(s,t) = e^{-st}$ as kernel gives us an especially important integral transform.
Let f be a function defined for $t \ge 0$. Then the integral

$$\mathcal{L}[f(t)] = \mathcal{L}[y] = \int_0^\infty e^{-st} f(t)dt \qquad (2)$$

is said to be the **Laplace transform** of f, provided that the integral converges.

Example 1: evaluate $\mathcal{L}[1]$.

Example 2: Evaluate $\mathcal{L}[t]$.

Example 3: Evaluate $\mathcal{L}[e^{-3t}]$

Example 4: Evaluate (using the Laplace table)

a) $\mathcal{L}[\sin(2t)]$

b) $\mathcal{L}[13\cos(5t)]$

c) $\mathcal{L}[13t\cos(5t)]$

d) $\mathcal{L}[6e^{-5t} + e^{3t} + 5t^3 - 9]$

e) $\mathcal{L}[e^{3t} + \cos(6t) - e^{2t}\sin(6t)]$

Inverse Laplace Transforms

We will now investigate how the Laplace transform can be used to solve certain types of equations for an unknown function. We begin with the concept of the inverse Laplace transform or, more precisely, the inverse of a Laplace transform $\mathcal{L}[y] = F(s)$.

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}[y] = F(s)$, we then say $f(t) = \mathcal{L}^{-1}\{F(s)\}$ or $y = \mathcal{L}^{-1}\{\mathcal{L}[y]\}$. For example,

Transform	Inverse Transform
$\mathcal{L}[1] = \frac{1}{s}$	$1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$
$\mathcal{L}[t] = \frac{1}{s^2}$	$t = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$
$\mathcal{L}[e^{-3t}] = \frac{1}{s+3}$	$e^{-3t} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

We shall see shortly (next section) how we can use Laplace transforms (and inverse Laplace transforms) to solve differential equations. We will see that we don't explicitly solve for an unknown function $f(t)$. Rather, we are able to solve for the Laplace transform $F(s)$ of $f(t)$ and then use inverse Laplace transforms to solve for $f(t)$. For example, if we suppose $F(s) = \frac{-2s+6}{s^2+4}$ is the Laplace transform of a function $f(t)$ and we wish to solve for $f(t)$, meaning find $f(t)$ such that $\mathcal{L}[f(t)] = F(s)$, we can use $f(t) = \mathcal{L}^{-1}\{F(s)\}$. We will see how to solve this problem in the coming examples.

Theorem 1. The following are some inverse transforms.

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad n = 1,2,3,\ldots$

(c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

When we evaluate inverse transforms, it often happens that the function of s under consideration does not match exactly the form of the Laplace transform $F(s)$ given in a table above. It may be necessary to rearrange the function of s by multiplying and dividing by an appropriate constant.

Example 5 (Inverse transform from the table): Evaluate

a) $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$

c) $\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$

d) $\mathcal{L}^{-1}\left\{\frac{1}{4s+1}\right\}$

Note: The inverse Laplace transform is also a linear transform, that is, for constant α and β

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}$$

where F and G are the transform of some functions f and g.

e) $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$

Example 6 (Inverse transform by completing the square):

Find the inverse Laplace transform of $F(s) = \frac{s+3}{s^2+6s+10}$

Example 7 (Inverse transform by partial fractions): Evaluate $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+6s+8}\right\}$

Example 8: Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

Extra practice questions:

1. Find the Laplace transforms:

a) $\mathcal{L}[t^3 e^{-2t}]$

b) $\mathcal{L}[e^{-2t-5}]$

c) $\mathcal{L}[4t^2 - 5\sin(3t)]$

2. Find the inverse Laplace transform:

a) $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

b) $\mathcal{L}^{-1}\left\{\frac{5}{s^2+16}\right\}$

c) $\mathcal{L}^{-1}\left\{\frac{3}{3s-2}\right\}$

d) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\}$

e) $\mathcal{L}^{-1}\left\{\frac{s-5}{s^2+1}\right\}$

f) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s}\right\}$

g) $\mathcal{L}^{-1}\left\{\frac{s+6}{s^2+6s+9}\right\}$

Solving Differential Equations by Laplace Transform

The main reason to use Laplace transforms is to solve differential equations. To that end we need to evaluate quantities such as $\mathcal{L}[y']$ and $\mathcal{L}[y'']$. For example, if f' is continuous for $t \geq 0$, then integration by parts gives

$$\mathcal{L}[y'] = \int_0^\infty e^{-st} f'(t)dt$$

(by parts with $u = e^{-st}$ and $dv = f'(t)$)

$$\int_0^\infty e^{-st} f'(t)dt = e^{-st}f(t)|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt$$

$$= e^{-st}f(t)|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)dt$$

therefore $\mathcal{L}[y'] = -f(0) + s\int_0^\infty e^{-st} f(t)dt = -f(0) + s\mathcal{L}(y)$

So,

$$\mathcal{L}[y'] = s\mathcal{L}[y] - f(0) \quad (3)$$

Here we have assumed that $e^{-st}f(t) \to 0$ as $t \to \infty$. Similarly, with the aid of (3), we have

$$\mathcal{L}[y''] = \int_0^\infty e^{-st} f''(t)dt$$

(by parts with $u = e^{-st}$ and $dv = f''(t)$)

$$\mathcal{L}[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt = e^{-st}f'(t)|_0^\infty - \int_0^\infty (-s)e^{-st}f'(t)\,dt$$

$$\mathcal{L}[f''(t)] = e^{-st}f'(t)|_0^\infty + s\int_0^\infty e^{-st}f'(t)dt$$

$$\mathcal{L}[f''(t)] = -f'(0) + s\int_0^\infty e^{-st}f'(t)dt = -f'(0) + s[s\mathcal{L}[y] - f(0)]$$

So,

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sy(0) - y'(0) \quad (4)$$

In like manner it can be shown that

$$\mathcal{L}[y'''] = s^3\mathcal{L}[y] - s^2y(0) - sy'(0) - y''(0) \quad (5)$$

The recursive nature of the Laplace transform of the derivatives of the function f, as it appears from (3), (4) and (5) gives the Laplace transform of the nth order of f:

$$\mathcal{L}\left[y^{(n)}\right] = s^n\mathcal{L}[y] - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)$$

Example 9: Solve the following initial value problems using Laplace Transforms

a) $2y' - 3y = 0, y(0) = -1$

b) $-9y'' - 4y = 0, y(0) = 1, y'(0) = 0$

c) $y'' + y = \cos t\,, y(0) = 1, y'(0) = 2$

d) Solve $y'' - 3y' + 2y = e^{-4t}, y(0) = 1, y'(0) = 5$.

Example 10: Use the Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t\,,\;\; y(0) = 6$$

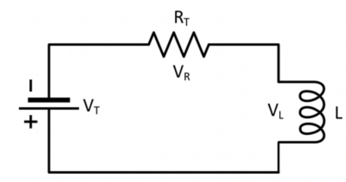
Applications

Electrical Current

Example 11. The general equation relating the current, voltage and an inductor of a simple electric circuit is

$$L\frac{d_i}{dt} + Ri = V.$$

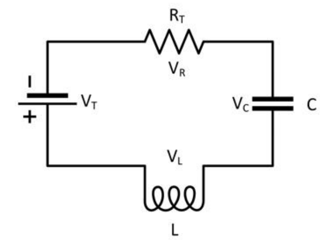
If $L = 1\ H, R = 10\ \Omega$ and $V = 6\ V$, find the current as a function of time knowing the initial current in the circuit is zero.



Example 12. An electric circuit contains a $0.1\ H$ inductor, a $0.625\ F$ capacitor, a voltage source of $10\cos 4t$, and negligible resistance (assume $R = 0$). If the initial charge on the capacitor is zero, and the initial current is 2 amps, find the current in the charge as a function of time t.

Recall that the general equation relating the current i (amps, A), the voltage V_T (volts, V), an inductance L (henry, H) and a capacitance (farad, F) of an electrical circuit in series is

$$V_T = L\frac{d^2Q}{dt^2} + R_T\frac{dQ}{dt} + \frac{Q}{C}$$



where Q represents the electric charge (in coulombs, C) and t represents the time (in seconds).

Simple Harmonic Motion

Example 13. A spring is stretched $1\ m$ from equilibrium by a weight of $4.9\ N$ (mass of $0.5\ kg$). The medium resists the motion with a force of $4v$, where v is the velocity of the motion. The differential equation describing the displacement of the weight is

$$\frac{1}{2}\frac{d^2y}{dt^2}+4\frac{dy}{dt}+16y=0.$$

If the initial velocity is $0\ m/s$, find an equation for its position at any time t.

Answers to Examples

1. $\mathcal{L}[1] = \frac{1}{s}$

2. $\mathcal{L}[t] = \frac{1}{s^2}$

3. $\mathcal{L}[e^{-3t}] = \frac{1}{s+3}$

4. a) $\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}$ b) $\mathcal{L}[13\cos(5t)] = \frac{13s}{s^2+25}$ c) $\mathcal{L}[13t\cos(5t)] = 13 \cdot \frac{s^2-25}{(s^2+25)^2}$

d) $\mathcal{L}[6e^{-5t} + e^{3t} + 5t^3 - 9] = \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$

e) $\mathcal{L}[e^{3t} + \cos(6t) - e^{2t}\sin(6t)] = \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{6}{(s-2)^2+36}$

5. a) $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{24}$ b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t$ c) $\mathcal{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} = 10\cos(4t)$

d) $\mathcal{L}^{-1}\left\{\frac{1}{4s+1}\right\} = \frac{1}{4}e^{-\frac{1}{4}t}$ e) $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2\cos(3t) - 2\sin(3t)$

6. $\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\cos(t)$

7. $\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+4s+13}\right\} = \frac{2}{3}e^{-4t} + \frac{1}{3}e^{-2t}$

8. $\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$

Extra practice questions

1. a) $\frac{6}{(s+2)^4}$ b) $\frac{e^{-5}}{s+2}$ c) $\frac{8}{s^3} - \frac{15}{s^2+9}$

2. a) e^{-3t} b) $\frac{5}{4}\sin(4t)$ c) $e^{\frac{2}{3}t}$ d) $\frac{1}{8}t\sin(4t)$ e) $\cos t - 5\sin t$

f) $\frac{1}{2}t + \frac{1}{2}e^{-\frac{t}{2}}$ g) $e^{-3t} + 3te^{-3t}$

9. a) $y(t) = -e^{\frac{3}{2}t}$ b) $y(t) = \cos\left(\frac{2}{3}t\right)$

c) $y(t) = \frac{1}{2}t\sin(t) + \cos(t) + 2\sin(t)$ d) $y(t) = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$

10. $y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$

11. $i(t) = \frac{3}{5}(1 - e^{-10t})$

12. $Q(t) = \frac{25}{2}t\sin(4t) + \frac{1}{2}\sin(4t)$

13. $y(t) = e^{-4t}\cos(4t) + e^{-4t}\sin(4t)$