

Expansion of Functions in Series

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Expansion of Functions in Series

Infinite Series

A **sequence** $\{a_n\}$ consists of an ordered family of real numbers a_n , one for each $n \in \mathbb{N}$. Thus a sequence looks like $a_1, a_2, \dots, a_n, a_{n+1}, \dots$. The number a_n is called the n -th term of the sequence.

Example 1. Find the first four terms of the sequences below.

a) $\{2^n\}_{n=0}^{\infty}$

b) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

c) $\{(-1)^n\}_{n=1}^{\infty}$

If $\{a_n\}_{n=1}^{\infty}$ is a sequence, then $\sum_{n=1}^{\infty} a_n$ denotes the infinite sum: $a_1 + a_2 + a_3 + \dots$. How can we add (sum) an infinite number of numbers? We can't!! But we can always find the sum of the finite number of numbers (partial sum). If we write the partial sum as S_n , then $S_n = \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n$. This creates another sequence S_0, S_1, S_2, \dots , and if this sequence converges (ie. $\lim_{n \rightarrow \infty} S_n = S$), then

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = \sum_{k=1}^{\infty} a_k = S.$$

In this case, we say the series has a finite sum S and is said to be **convergent**, if the sequence of the partial sums is divergent, then the series is said to be **divergent**.

Example 2. Do the following series converge or diverge?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(b) $\sum_{n=1}^{\infty} 3^n$

(c) $\sum_{n=1}^{\infty} (-1)^n$

Geometric Series

Fix a number $a \neq 0$. Build a sequence by multiplying the previous term with r (a fixed number) and starting with a . We get $a, ar, ar^2, ar^3, \dots = \{ar^n\}_{n=0}^{\infty}$. What does the partial sum equal?

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

ONLY if $|r| < 1$.

Example 3. Do the following series converge or diverge? If they converge, find the sum.

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^n}$$

$$\sum_{n=0}^{\infty} 2^{4n} 3^{-n}$$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{5^{n+2}}$$

$$\sum_{n=0}^{\infty} (-1)^{n+2} \frac{2^{2n}}{5}$$

Power series

Power series centered at zero

Power series are of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

The convergence of the power series may depend on the value of x and the coefficients c_n .

Interval of convergence: Those values of x for which the series converges (always include 0). Power series define a function whose domain is the set of x for which it converges.

Recall: Geometric series: If for all n , $c_n = 1$, then we have

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{for } |x| < 1,$$

and diverges for $|x| \geq 1$.

Maclaurin series

Maclaurin series centered at $x = 0$

On its interval of convergence, a power series $\sum_{n=0}^{\infty} c_n x^n$ represents a function. This representation is *unique* for a given center. If

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

Then

$$\begin{aligned} f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots \\ f''(x) &= 1 \cdot 2c_2 + 2 \cdot 3c_3 x + 3 \cdot 4c_4 x^2 + \dots \\ f'''(x) &= 1 \cdot 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4 x + \dots \\ f^{(4)}(x) &= 1 \cdot 2 \cdot 3 \cdot 4c_4 + \dots \end{aligned}$$

So if we let $c_0 = f(0)$, $c_1 = f'(0)$, $c_2 = \frac{f''(0)}{2}$, $c_3 = \frac{f^{(3)}(0)}{6}$, ..., $c_n = \frac{f^{(n)}(0)}{n!}$, then

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Maclaurin Series

If $a = 0$ (center is at 0), then we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

This is a **Maclaurin Series** of $f(x)$.

Example 5. Find the first three nonzero terms of the Maclaurin series expansion of the function

$$f(x) = \sin(x)$$

Some special Maclaurin series

Here is a list of functions whose Maclaurin series are sufficiently regular:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots \quad (\text{geometric series})$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Operations with Series

We have currently worked with the idea that a power series $\sum_{n=0}^{\infty} c_n x^n$ defines a function on its interval of convergence. Now let's look at this from another side: ie. Given a function $f(x)$, is there a power series representation for it?

Example 7. Consider the function $f(x) = \frac{1}{1-x}$. What can we say about $g(x) = \frac{x^4}{1-x}$?

Example 8. Find the Maclaurin series of e^{x^2} and $x^2 e^x$ (without having to compute all the derivatives).

Example 9. Write the Maclaurin series expansion for the following functions. Only include the first three nonzero terms.

a) $x \cos x$

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b) $\sin(x^2)$

c) $x^2 e^{-3x}$

d) $3x \ln(1 - 2x)$

e) $\frac{5x}{3+x}$

Differentiation and Integration of Power Series

If a power series $\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence $R > 0$ and if

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots = \sum_{n=0}^{\infty} c_n x^n.$$

Then

$$(a) f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

$$= \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$(b) \int f(x) dx = c + c_0 x + \frac{1}{2} c_1 x^2 + \frac{1}{3} c_2 x^3 + \dots$$

$$= c + \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1}$$

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Example 10.

(a) State the Maclaurin series of e^{x^2} . Include the first three nonzero terms.

(b) Using part (a), calculate $\int e^{x^2} dx$ and $\int_0^1 e^{x^2} dx$.

Example 11. Find the Maclaurin series for $\cos x^2$ and $\int_0^1 \cos(x^2) dx$.

Computations by Use of Series Expansions

Power series can also be used to compute numerical values of functions. By including sufficient terms, we can calculate a value to a certain degree of accuracy. These are approximations and the more terms you keep, the better your approximation. Through these calculations, decimal approximations of e and π can be calculated and programmed into a calculator or computer.

Example 12. Calculate the value of $e^{0.2}$. Use the first three terms of a Maclaurin series of e^x to evaluate it, and then compare this result with the actual value. Then do it again keeping the first 4 terms (and then 5 terms and 6 terms) of the Maclaurin series and compare it with the actual value. What do you notice?

Example 13. Write the first 3 nonzero terms of the Maclaurin series of $\sin(2x)$ then calculate the value of $\sin\left(\frac{2\pi}{5}\right)$. Compare this answer to the actual value.

Introduction to Fourier Series

Many problems encountered in various fields of science and technology involve **periodic functions** (examples are alternating-current voltages and mechanical oscillations). Recall that a periodic function with period P has the property $f(x + P) = f(x)$. We will now look at series made up of terms of sines and cosines.

Fourier of Sines and Cosines: Period = 2π

First, we will assume that a function $f(x)$ may be represented by a series of sines and cosines

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots \\ + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots \quad (1)$$

Even though some of the individual trigonometric terms have period less than 2π (example $\cos 2x$ has period π), all terms repeat every 2π units. This series is called a **Fourier series**.

The goal is to find the coefficients a_n and b_n . We can find them by multiplying all the terms of $f(x)$ by $\cos mx$ and then using certain properties of integrals from $-\pi$ to π (taking advantage of the period 2π). We have

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots) (\cos mx) \, dx \\ + \int_{-\pi}^{\pi} (b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots) (\cos mx) \, dx \quad (2)$$

Using integration techniques, we can find the values of the coefficients. For a_n (where $n \neq m$), we get

$$\int_{-\pi}^{\pi} a_0 \cos mx \, dx = \frac{a_0}{m} \sin mx \Big|_{-\pi}^{\pi} = \frac{a_0}{m} (0 - 0) = 0 \\ \int_{-\pi}^{\pi} a_n \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} a_n \frac{\cos(nx + mx) + \cos(nx - mx)}{2} \, dx \\ = \frac{a_n}{2} \left[\frac{\sin(nx + mx)}{n + m} + \frac{\sin(nx - mx)}{n - m} \right] \Big|_{-\pi}^{\pi} = 0 \quad (n \neq m)$$

All these values are equal to zero since the sine of any multiple of π is zero.

If we consider when $n = m$, we have

$$\int_{-\pi}^{\pi} a_n \cos nx \cos nx \, dx = \int_{-\pi}^{\pi} a_n \cos^2 nx \, dx = \int_{-\pi}^{\pi} a_n \frac{(\cos 2x + 1)}{2} \, dx = \frac{a_n}{2} \left[\frac{\sin 2x}{2} + x \right] \Big|_{-\pi}^{\pi} \\ = \frac{a_n}{2} [(0 + \pi) - (0 - \pi)] = \pi a_n \quad (3)$$

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For b_n (where $n \neq m$), we get

$$\begin{aligned} \int_{-\pi}^{\pi} b_n \sin nx \cos mx \, dx &= \int_{-\pi}^{\pi} b_n \frac{\sin(nx + mx) + \sin(nx - mx)}{2} \, dx \\ &= -\frac{b_n}{2} \left[\frac{\cos(nx + mx)}{n + m} + \frac{\cos(nx - mx)}{n - m} \right] \Big|_{-\pi}^{\pi} \\ &= -\frac{b_n}{2} \left[\left(\frac{\cos(n + m)\pi}{n + m} + \frac{\cos(n - m)\pi}{n - m} \right) - \left(\frac{\cos(n + m)(-\pi)}{n + m} + \frac{\cos(n - m)(-\pi)}{n - m} \right) \right] \\ &= 0 \end{aligned}$$

since $\cos(-\theta) = \cos \theta$. Now if we consider $n = m$, we have

$$\begin{aligned} \int_{-\pi}^{\pi} b_n \sin nx \cos nx \, dx &= \int_{-\pi}^{\pi} b_n \frac{\sin(2nx) + \sin(0)}{2} \, dx = \int_{-\pi}^{\pi} \frac{b_n}{2} \sin(2nx) \, dx \\ &= -\frac{b_n}{2} \left[\frac{\cos 2nx}{2n} \right] \Big|_{-\pi}^{\pi} = -\frac{b_n}{2} \left[-\frac{1}{2n} + \frac{1}{2n} \right] = 0 \end{aligned}$$

We have that all the integrals equal zero except for the case of a_n when $n = m$. Using this information in our original integral and using equation 3, we have

$$\int_{-\pi}^{\pi} a_n \cos nx \cos nx \, dx = \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \pi a_n$$

where the first inequality is true since all terms in the integral containing $f(x) \cos(nx)$ are equal to zero other than when $n = m$ for a_n . In detail, we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx \, dx &= \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots)(\cos nx) \, dx \\ &\quad + \int_{-\pi}^{\pi} (b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots)(\cos nx) \, dx \\ &= \int_{-\pi}^{\pi} (a_n \cos nx)(\cos nx) \, dx = \int_{-\pi}^{\pi} a_n \cos^2 nx \, dx = a_n \pi \end{aligned}$$

Since $a_n \pi = \int_{-\pi}^{\pi} f(x) \cos nx \, dx$, we have that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

Expansion of Functions in Series

This equation allows us to find all values of a_n in our Fourier series other than a_0 . To find a_0 , we use equation (1) and integrate from $-\pi$ to π . All the terms with sines and cosines are equal to zero since $\sin n\pi = 0$ and $\cos(-n\pi) = -\cos n\pi$ so they all cancel out other than a_0 . We get

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} a_0 dx = a_0 [x]_{-\pi}^{\pi} = a_0(\pi + \pi) = 2\pi a_0.$$

And so

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

Similarly, we can multiply equation (1) by $\sin mx$ and then integrate from $-\pi$ to π to solve for b_n . We get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

In summary:

Fourier series (Period 2π)

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots \\ + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots \quad (4)$$

with

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

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Example 16. Find the Fourier series for the square wave function

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

- a. Find the missing Fourier coefficients. Give your answers either in exact form or rounded to 1 decimal place.

a_0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
		0		0		0		0

- b. Use the Fourier coefficients found in (a) to write the corresponding Fourier series.

More About Fourier Series

The effort used to evaluate the definite integrals that define the coefficients a_0 , a_n and b_n in the expansion of a function f in a Fourier series is significantly reduced when f is either an even or an odd function.

Even and Odd Functions

A function f is said to be

$$\text{even if } f(-x) = f(x) \quad \text{and odd if } f(-x) = -f(x)$$

On a symmetric interval such as $(-p, p)$, the graph of an even function is symmetric with respect to the y -axis, whereas the graph of an odd function is symmetric with respect to the origin. For example,

$$\begin{aligned} f(x) = x^2 \text{ is even since } f(-x) &= (-x)^2 = x^2 = f(x) \\ f(x) = x^3 \text{ is odd since } f(-x) &= (-x)^3 = -x^3 = -f(x) \end{aligned}$$

See Figures 1 and 2.

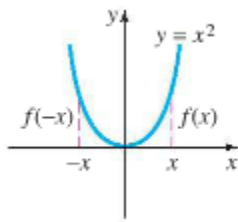


Figure 1: Even function; graph symmetric with y -axis

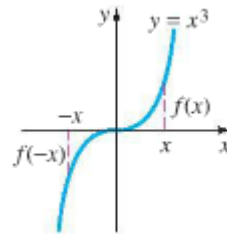


Figure 2: Odd function; graph symmetric with respect to the origin

The trigonometric cosine and sine functions are even and odd, respectively, since

$$\cos(-x) = \cos x \quad \text{and} \quad \sin(-x) = -\sin x.$$

The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither odd nor even.

Properties of Even/Odd functions

1. The product of two even functions is even.
2. The product of two odd functions is even.
3. The product of an even and an odd function is odd.
4. The sum (difference) of two even functions is even.
5. The sum (difference) of two odd functions is odd.
6. If f is even, then $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$.
7. If f is odd, then $\int_{-a}^a f(x)dx = 0$.

Therefore, if f is an even function on $(-\pi, \pi)$, the integrand from equation (4) is odd, and the coefficients of a Fourier series give $b_n = 0$. If f is an odd function on $(-\pi, \pi)$, the integrand from equation (4) is even, and the coefficients of a Fourier series give $a_n = 0$.

Fourier Series with Period $2L$

If f is a function on $(-L, L)$, since $\sin \frac{n\pi}{L}(x + 2L) = \sin n \left(\frac{\pi x}{L} + 2\pi \right) = \sin \frac{n\pi x}{L}$, the coefficients of a Fourier series

$$f(x) = a_0 + a_1 \cos \frac{\pi x}{L} + a_2 \cos \frac{2\pi x}{L} + \dots + a_n \cos \frac{n\pi x}{L} + \dots \\ + b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + \dots + b_n \sin \frac{n\pi x}{L} + \dots \quad (5)$$

are

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

If f is an even function on $(-L, L)$, then using the properties of even/odd functions, the coefficients of a Fourier series are

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{L} \int_0^L f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \\ \text{(because } f(x) \cos \frac{n\pi x}{L} \text{ is even)}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx = 0 \\ \text{because } f(x) \sin \frac{n\pi x}{L} \text{ is odd}$$

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Similarly, when f is odd on the interval $(-L, L)$,

$$a_n = 0, \quad n = 0, 1, 2, \dots, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

We summarize the results in the following definition.

Fourier Cosine and Sine Series

- (i) The Fourier series of an even function f defined on the interval $(-L, L)$ is the **cosine series**

$$f(x) = a_0 + a_1 \cos \frac{\pi x}{L} + a_2 \cos \frac{2\pi x}{L} + \dots + a_n \cos \frac{n\pi x}{L} + \dots = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$\text{where } a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

- (ii) The Fourier series of an odd function f defined on the interval $(-L, L)$ is the **sine series**

$$f(x) = b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + \dots + b_n \sin \frac{n\pi x}{L} + \dots = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\text{where } b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Expansion of Functions in Series

Example 17. Find the Fourier series for the square wave function

$$f(x) = \begin{cases} -3, & -2 \leq x < 0 \\ 3, & 0 \leq x < 2 \end{cases}$$

- a.** Sketch the graph. Is this an odd or even function?
b. Find the missing Fourier coefficients. Give your answers either in exact form or rounded to 1 decimal place.

a_0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0		0		0		0		0

- c.** Use the Fourier coefficients found in **(b)** to write the corresponding Fourier series.

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Example 18. Find the Fourier series for the function

$$f(x) = \begin{cases} -2 & -\pi \leq x \leq \frac{-\pi}{2}, \frac{\pi}{2} \leq x \leq \pi \\ 2 & \frac{-\pi}{2} \leq x \leq \frac{\pi}{2} \end{cases}$$

- a.** Sketch the graph. Is this an odd or even function?
b. Find the missing Fourier coefficients. Give your answers either in exact form or rounded to 1 decimal place.

a_0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0		0		0		0		0

- c.** Use the Fourier coefficients found in **(b)** to write the corresponding Fourier series.

Answers to Examples

1. a) 1,2,4,8,16,32, ... b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ c) -1, +1, -1, +1, -1, +1, ...
 2. a) series converges b) series diverges c) series diverges
 3. a) series converges to $\frac{3}{2}$ b) series converges to $\frac{21}{4}$ c) series diverges
 d) series converges to $-\frac{1}{35}$ e) series diverges
 4. $\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$
 5. $g(x) = x^4 + x^5 + x^6 + x^7 + \dots$
 6. $x^2 e^x = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \dots$
 7. a) $x \cos x = x - \frac{x^3}{2} + \frac{x^5}{24} - \dots$ b) $\sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \dots$
 c) $x^2 e^{-3x} = x^2 - 3x^3 + \frac{9x^4}{2} - \frac{27x^5}{6} + \dots$ d) $3x \ln(1 - 2x) = -6x - 6x^3 - 8x^4 - \dots$
 e) $\frac{5x}{3+x} = \frac{5x}{3} - \frac{5x^2}{9} + \frac{5x^3}{27} + \dots$
 8. a) $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \dots$ $\int_0^1 e^{x^2} dx = 1.433$
 b) $\int e^{x^2} dx = x + \frac{x^3}{3} + \frac{x^5}{10} + \dots$ $\int_0^1 \cos(x^2) dx = 0.9$
 9. $\cos(x^2) = 1 - \frac{x^4}{2} + \frac{x^6}{24} - \dots$
 10.

3 terms	$e^{0.2} \approx 1 + 0.2 + \frac{0.2^2}{2} = 1.22$
4 terms	$e^{0.2} \approx 1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{6} = 1.2213333$
5 terms	$e^{0.2} \approx 1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{6} + \frac{0.2^4}{24} = 1.2214$
6 terms	$e^{0.2} \approx 1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{6} + \frac{0.2^4}{24} + \frac{0.2^5}{120} = 1.221402667$

11. $\sin \frac{2\pi}{5} \approx 0.95201707$

12. a)

a_0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	0	0	0	0	$\frac{4}{\pi}$	0	$\frac{4}{3\pi}$	0

b) $f(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \dots$

13. a) Function is odd

b)

a_0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	0	0	0	0	$\frac{12}{\pi}$	0	$\frac{4}{\pi}$	0

c) $f(x) = \frac{12}{\pi} \sin x + \frac{4}{\pi} \sin 3x + \dots$

14. a) Function is even

b)

a_0	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	$\frac{8}{\pi}$	0	$-\frac{8}{3\pi}$	0	0	0	0	0

c) $f(x) = \frac{8}{\pi} \cos x - \frac{8}{3\pi} \cos 3x + \frac{8}{5\pi} \cos 5x - \dots$